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Research Article

Thermodynamic Relations for Kiselev and Dilaton Black Hole

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We investigate the thermodynamics and phase transition for Kiselev black hole and dilaton black hole. Specifically we consider Reissner-Nordström black hole surrounded by radiation and dust and Schwarzschild black hole surrounded by quintessence, as special cases of Kiselev solution. We have calculated the products relating the surface gravities, surface temperatures, Komar energies, areas, entropies, horizon radii, and the irreducible masses at the Cauchy and the event horizons. It is observed that the product of surface gravities, product of surface temperature, and product of Komar energies at the horizons are not universal quantities for the Kiselev solutions while products of areas and entropies at both the horizons are independent of mass of the above-mentioned black holes (except for Schwarzschild black hole surrounded by quintessence). For charged dilaton black hole, all the products vanish. The first law of thermodynamics is also verified for Kiselev solutions. Heat capacities are calculated and phase transitions are observed, under certain conditions.

1. Introduction

Black holes are the most exotic objects in physics and their connection with thermodynamics is even more surprising. Discovery of Hawking radiations leads to the identification of black holes as thermodynamic objects with physical temperature and entropy. Once black holes are identified as thermodynamical objects, it is quite natural to find whether they also behave as familiar thermodynamic systems. The analogy between the black hole thermodynamics and four laws of thermodynamics was first proposed in the 1970s [1–6]. The important results of the black hole thermodynamics are the association of temperature (T) and entropy (S) with surface gravity (κ) and area (A) of the black hole event horizon, respectively. Laws of black hole thermodynamics are studied in the literature [7–10]. In [11] universal properties of black holes and the first law of black hole inner mechanics are discussed. In [12] horizon entropy sums in A(dS) spacetimes are studied. In [13] the authors have discussed the spin entropy of a rotating black hole. The study of phase transition in black holes is a fascinating topic. The phenomenon of phase transition in black hole thermodynamics was first observed long ago [14–16]. If a black hole has a Cauchy horizon ($\mathscr{H}^-$) and an event horizon($\mathscr{H}^+$), then it is quite interesting to study different quantities like the product of areas of a black hole on these horizons.

In [17–19] the thermal products for rotating black holes are studied. In [20] area products for stationary black hole horizons are calculated. The calculations show that sometimes these products depend not only on the ADM (Arnowitt-Deser-Misner) mass parameter but also on the charge and angular momentum. The relations that are independent of the black hole mass are of particular interest because these may turn out to be universal and hold for more general solutions with nontrivial surroundings too.

Kiselev [21] considered Einstein's field equation surrounded by quintessential matter and proposed new solutions, dependent on state parameter ω of the matter surrounding black hole. Recently, some dynamical aspects, that is, collision between particles and their escape energies after collision around Kiselev black hole [22], have been studied. In this work we consider the solution of Reissner-Nordström

(RN) black hole surrounded by energy-matter, derived by Kiselev, and study the important thermodynamic features of black hole at the horizons and generalize some already existing results for Cauchy horizon. We also consider the solution of Schwarzschild black hole surrounded by energy-matter and analyzed its different thermodynamic products. Furthermore, we have considered the charged dilaton black hole and computed its various thermodynamic products.

The plan of the work is as follows. In Section 2, we discuss the basic aspects of RN black hole surrounded by radiation. We have shown that products of surface gravity, temperature, and Komar energy calculated on the inner and outer horizons are not universal due to dependance on mass of the black hole, while products of horizons and products of area and entropy calculated at $\mathscr{H}^{\pm}$ are independent of mass of black hole. In Section 2.1, the Smarr formula for the black hole is derived and, using the obtained expression of mass, calculations for the first law of thermodynamics are given there. In Section 2.2, we have discussed the irreducible mass and the rest mass is written in terms of irreducible mass. In Section 2.3, heat capacity of black hole is calculated, and phase transition is discussed. In Section 3, the metric of RN black hole surrounded by dust is discussed and some basic aspects of black hole thermodynamics are discussed. In Sections 3.1, 3.2, and 3.3, we study the irreducible mass, Smarr formula, and heat capacity of the black hole, respectively. In Section 4, the metric of Schwarzschild black hole surrounded by quintessence is discussed. In Section 4.1, the Smarr formula and first law of thermodynamics are discussed, irreducible mass is studied in Section 4.2, and analysis of heat capacity of the black hole is completed in Section 4.3. In Section 5, we compute thermodynamic product relations for dilaton black hole. In the last section we concluded the work. We use units in which $G = \hbar = c = 1$.

2. RN Black Hole Surrounded by Radiation

The spherically symmetric and static solutions for Einstein's field equations, surrounded by energy-matter, as investigated by Kiselev [21] can be written as

$$ds^2 = -f(r)\,dt^2 + \frac{1}{f(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \quad (1)$$

where

$$f(r) = 1 - \frac{2\mathscr{M}}{r} + \frac{Q^2}{r^2} - \frac{\sigma}{r^{3\omega+1}}; \quad (2)$$

here $\mathscr{M}$ is mass of the black hole, Q is the electric charge, σ is normalization parameter, and ω is the state parameter of the matter around black hole. We consider the cases when RN black hole is surrounded by radiation ($\omega = 1/3$) and dust ($\omega = 0$). For $\omega = 1/3$ two horizons of the black hole are obtained from

$$1 - \frac{2\mathscr{M}}{r} + \frac{Q^2}{r^2} - \frac{\sigma_r}{r^2} = 0; \quad (3)$$

that is,

$$r_{\pm} = \mathscr{M} \pm \sqrt{\mathscr{M}^2 - Q^2 + \sigma_r}. \quad (4)$$

Here σ_r denotes the normalization parameter for radiation case, with dimensions, $[\sigma_r] = L^2$, where L denotes length, r_+ is the outer horizon named as event horizon $\mathscr{H}^+$, and r_- is the inner horizon known as Cauchy horizon $\mathscr{H}^-$. Cauchy horizon is a null surface of infinite blue-shift, while the event horizon is an infinite red-shift surface [23]. The product of the two horizons,

$$r_+ r_- = Q^2 - \sigma_r, \quad (5)$$

is independent of the mass of the black hole but depends on electric charge and σ_r. Areas of two horizons of the black hole are

$$\begin{aligned}\mathscr{A}_{\pm} &= \int_0^{2\pi}\int_0^{\pi}\sqrt{g_{\theta\theta}g_{\phi\phi}}\,d\theta\,d\phi = 4\pi r_{\pm}^2 \\ &= 4\pi\left(2\mathscr{M}r_{\pm} - Q^2 + \sigma_r\right).\end{aligned} \quad (6)$$

The corresponding semiclassical Bekenstein-Hawking entropy at the horizons $\mathscr{H}^{\pm}$ is [24]

$$\mathscr{S}_{\pm} = \frac{\mathscr{A}_{\pm}}{4} = \pi\left(2\mathscr{M}r_{\pm} - Q^2 + \sigma_r\right). \quad (7)$$

Hawking temperature of the horizons is determined by using the formula

$$\begin{aligned}T_{\pm} &= \frac{1}{4\pi}\frac{df}{dr}\bigg|_{r=r_{\pm}} = \frac{1}{4\pi}\left[\frac{r_{\pm}^2 - Q^2 + \sigma_r}{r_{\pm}^3}\right] \\ &= \frac{r_{\pm} - \mathscr{M}}{2\pi\left(2\mathscr{M}r_{\pm} - Q^2 + \sigma_r\right)},\end{aligned} \quad (8)$$

where we have used $Q^2 = 2\mathscr{M}r_{\pm} + \sigma_r - r_{\pm}^2$.

Surface gravity is the acceleration due to gravity at the horizon of a black hole. It is defined as the force required to an observer at infinity, for holding a particle (of unit mass) in place at the event horizon, given as [25]

$$\begin{aligned}\kappa_{\pm} &= \frac{1}{2}\frac{df}{dr}\bigg|_{r=r_{\pm}} = 2\pi T_{\pm}, \\ \kappa_{\pm} &= \frac{r_{\pm} - \mathscr{M}}{\left(2\mathscr{M}r_{\pm} - Q^2 + \sigma_r\right)}.\end{aligned} \quad (9)$$

The Komar energy of the black hole given by [26]

$$E_{\pm} = 2\mathscr{S}_{\pm}T_{\pm} = r_{\pm} - \mathscr{M}. \quad (10)$$

Products of surface gravities and surface temperatures at $\mathscr{H}^{\pm}$ are

$$\kappa_+\kappa_- = 4\pi^2 T_+T_- = \frac{Q^2 - \sigma_r - \mathscr{M}^2}{\left(Q^2 - \sigma_r\right)^2}. \quad (11)$$

Product of the Komar energies at $\mathscr{H}^{\pm}$ is

$$E_+E_- = Q^2 - \mathscr{M}^2 - \sigma_r. \quad (12)$$

It is clear that all these products (except product of horizons r_+r_-) are depending on mass of the black hole, so these quantities are not universal. We also calculate the products of areas and entropies at $\mathscr{H}^{\pm}$, which turn out to

$$\mathscr{A}_{+}\mathscr{A}_{-} = 16\mathscr{S}_{+}\mathscr{S}_{-} = 16\pi^2\left(Q^2-\sigma_r\right)^2. \tag{13}$$

Note that both the products are independent of mass, so these are universal quantities.

2.1. Smarr Formula for Cauchy Horizon ($\mathscr{H}^{-}$). The expression for area of the black hole can be rewritten using the idea proposed by Smarr [27, 28] as

$$\mathscr{A} = 4\pi\left[2M^2-Q^2+\sigma_r+2M\sqrt{M^2-Q^2+\sigma_r}\right]. \tag{14}$$

The area of both horizons must be constant given by

$$\mathscr{A}_{\pm} = 4\pi\left[2M^2-Q^2+\sigma_r\pm 2M\sqrt{M^2-Q^2+\sigma_r}\right]. \tag{15}$$

Using (15) mass of the black hole or ADM mass is expressed in terms of the areas of horizons as follows:

$$\begin{aligned}\mathscr{M}^2 &= \frac{\mathscr{A}_{\pm}}{16\pi}+\frac{Q^4\pi}{\mathscr{A}_{\pm}}+\frac{Q^2}{2}-\frac{2\pi\sigma_r Q^2}{\mathscr{A}_{\pm}}-\frac{\sigma_r}{2}+\frac{\pi\sigma_r^2}{\mathscr{A}_{\pm}}\\ &= \frac{\mathscr{A}_{\pm}}{16\pi}+\frac{\pi Q^4}{\mathscr{A}_{\pm}}+Q^2\left(\frac{\mathscr{A}_{\pm}-4\pi\sigma_r}{2\mathscr{A}_{\pm}}\right)\\ &\quad-\sigma_r\left(\frac{\mathscr{A}_{\pm}-2\pi\sigma_r}{2\mathscr{A}_{\pm}}\right).\end{aligned} \tag{16}$$

Since the first law of thermodynamics states that change in mass of a black hole is related to change in its area and electric charge and also the effective surface tensions at the horizons are proportional to the temperatures of the black hole horizons, we can write:

$$d\mathscr{M} = \mathscr{T}_{\pm}d\mathscr{A}_{\pm}+\Phi_{\pm}dQ, \tag{17}$$

where $\mathscr{T}_{\pm}$ and $\mathscr{A}_{\pm}$ are the physical invariants of both horizons, defined as

$$\begin{aligned}\mathscr{T}_{\pm} &= \text{effective surface tension at horizons}\\ &= \frac{1}{\mathscr{M}}\left(\frac{1}{32\pi}-\frac{Q^4\pi}{2\mathscr{A}_{\pm}^2}+\frac{Q^2\sigma_r\pi}{\mathscr{A}_{\pm}^2}-\frac{\pi\sigma_r^2}{2\mathscr{A}_{\pm}}\right)\\ \Phi_{\pm} &= \text{electromagnetic potentials at horizons}\\ &= \frac{1}{\mathscr{M}}\left(\frac{2\pi Q^3}{\mathscr{A}_{\pm}}-\frac{2\pi Q\sigma_r}{\mathscr{A}_{\pm}}+\frac{Q}{2}\right).\end{aligned} \tag{18}$$

We can rewrite effective surface tension as

$$\begin{aligned}\mathscr{T}_{\pm} &= \frac{1}{32\pi\mathscr{M}}\left(1-\frac{16\pi^2\left(Q^4-2Q^2\sigma_r+\sigma_r^2\right)}{\mathscr{A}_{\pm}^2}\right)\\ &= \frac{1}{16\pi\mathscr{M}}\left(1-\frac{2M^2-Q^2+\sigma_r}{r_{\pm}^2}\right)\\ &= \frac{1}{8\pi}\left(\frac{r_{\pm}-\mathscr{M}}{2\mathscr{M}r_{\pm}-Q^2+\sigma_r}\right)=\frac{\kappa_{\pm}}{8\pi}.\end{aligned} \tag{19}$$

That is, RN black hole surrounded by radiation satisfies the first law of thermodynamics.

2.2. Christodoulou-Ruffini Mass Formula for RN Black Hole Surrounded by Radiation. Christodoulou and Christodoulou and Ruffini [1, 2] had shown that the mass of a black hole could be increased or decreased, but there is no way to decrease the irreducible mass $\mathscr{M}_{\text{irr}}$ of a black hole. In fact, most processes result in an increase in $\mathscr{M}_{\text{irr}}$ and during reversible process this quantity also does not change. Also, the surface area of a black hole has behavior [29]:

$$d\mathscr{A}_{\pm} \geq 0, \tag{20}$$

so there exists a relation between area and irreducible mass. $\mathscr{M}_{\text{irr}}$ is proportional to the square root of the black hole's area. Since the RN-radiation space-time has regular event horizon and Cauchy horizons, the irreducible mass of a black hole is proportional to the square root of its surface area [2]:

$$\sqrt{\frac{\mathscr{A}_{\pm}}{16\pi}} = \mathscr{M}_{\text{irr}\pm} = \sqrt{\frac{r_{\pm}^2}{4}}. \tag{21}$$

The irreducible mass defined on inner and outer horizons is $\mathscr{M}_{\text{irr}-}$ and $\mathscr{M}_{\text{irr}+}$, respectively. The product of the irreducible mass at the horizons $\mathscr{H}^{\pm}$ is

$$\mathscr{M}_{\text{irr}+}\mathscr{M}_{\text{irr}-} = \sqrt{\frac{\mathscr{A}_{+}\mathscr{A}_{-}}{(16\pi)^2}} = \frac{Q^2-\sigma_r}{4}. \tag{22}$$

This product is universal because it does not depend on the mass of the black hole. The expression for the rest mass of the rotating charged black hole given by Christodoulou and Ruffini in terms of its irreducible mass, angular momentum, and charge is [2]

$$\mathscr{M}^2 = \left(\mathscr{M}_{\text{irr}\pm}+\frac{Q^2}{4\mathscr{M}_{\text{irr}\pm}}\right)^2+\frac{J^2}{4\mathscr{M}_{\text{irr}\pm}^2}. \tag{23}$$

For RN black hole surrounded by radiation expression of mass in terms of irreducible mass becomes

$$\mathscr{M}^2 = \frac{\rho_{\pm}^4+Q^4}{4\rho_{\pm}^2}+\frac{\pi}{2}\left(\sigma_r^2-2\sigma_r Q^2\right)+2\mathscr{M}_{\text{irr}+}\mathscr{M}_{\text{irr}-}, \tag{24}$$

where $\rho_{\pm} = 2\mathscr{M}_{\text{irr}\pm}$.

2.3. Heat Capacity $C_{\pm}$ on $\mathscr{H}^{\pm}$. Another important measure to study the thermodynamic properties of a black hole is the heat capacity of black hole. The nature (positivity or negativity) of heat capacity reflects the change in the stability properties of the thermal system (black hole). A black hole with negative heat capacity is in unstable equilibrium state; that is, by emitting Hawking radiations, it may decay to a hot flat space or by absorbing a radiation it may grow without limit [30]. Heat capacity of a black hole is given by

$$C_{\pm} = \frac{\partial\mathscr{M}}{\partial T_{\pm}}, \tag{25}$$

where mass $\mathscr{M}$ in terms of $r_{\pm}$ is

$$\mathscr{M} = \frac{r_{\pm}^2+Q^2-\sigma_r}{2r_{\pm}}. \tag{26}$$

The partial derivatives of mass $\mathscr{M}$ and temperature $T_{\pm}$ with respect to r_{+} are

$$\frac{\partial \mathscr{M}}{\partial r_{\pm}} = \frac{r_{\pm}^{2} - Q^{2} + \sigma_{r}}{2r_{\pm}^{2}}, \tag{27}$$

and from (8) we have

$$\frac{\partial T}{\partial r_{\pm}} = \frac{1}{2\pi}\frac{\left(3Q^{2} - r_{\pm}^{2} - 3\sigma_{r}\right)}{r_{\pm}^{4}}. \tag{28}$$

The expression for heat capacity $C_{\pm} = (\partial \mathscr{M}/\partial r_{\pm})(\partial r_{\pm}/\partial T_{\pm})$ for RN black hole surrounded by radiation at horizons becomes

$$C_{\pm} = \frac{2\pi r_{\pm}^{2}\left(r_{\pm}^{2} - Q^{2} + \sigma_{r}\right)}{3Q^{2} - 3\sigma_{r} - r_{\pm}^{2}}. \tag{29}$$

Note that there are two possible cases for heat capacity to be positive.

Case 1. When both $r_{\pm}^{2} - Q^{2} + \sigma_{r}$ and $3Q^{2} - 3\sigma_{r} - r_{\pm}^{2}$ are positive.

Case 2. When both $r_{\pm}^{2} - Q^{2} + \sigma_{r}$ and $3Q^{2} - 3\sigma_{r} - r_{\pm}^{2}$ are negative.

Since we are interested in positive r only, so Case 1 implies that heat capacity is positive if

$$\sqrt{Q^{2} - \sigma_{r}} < r < \sqrt{3\left(Q^{2} - \sigma_{r}\right)}; \tag{30}$$

from Case 2 we get

$$3\left(Q^{2} - \sigma_{r}\right) < r^{2} < \left(Q^{2} - \sigma\right), \tag{31}$$

which is not possible, so we exclude this case.

Heat capacity is negative if the following cases hold.

Case a. $r_{\pm}^{2} - Q^{2} + \sigma_{r} > 0$ and $3Q^{2} - 3\sigma_{r} - r_{\pm}^{2} < 0$.

Case b. $r_{\pm}^{2} - Q^{2} + \sigma_{r} < 0$ and $3Q^{2} - 3\sigma_{r} - r_{\pm}^{2} > 0$.

Case *a* implies that for

$$r > \sqrt{3Q^{2} - 3\sigma_{r}}, \tag{32}$$

heat capacity is negative. From Case *b* we get negative capacity for

$$-\sqrt{Q^{2} - \sigma} < r < \sqrt{Q^{2} - \sigma}; \tag{33}$$

we are interested in positive r only. The region where heat capacity is negative corresponds to an instable region around black hole, whereas a region in which the heat capacity is positive represents a stability region. Behavior of heat capacity given in (29) is shown in Figure 1. Heat capacity is negative in the regions $0 < r < 0.4898$ and $r > 0.8485$, while positive in $0.4898 < r < 0.8485$. Interestingly, the product of heat capacity on $\mathscr{H}^{\pm}$ becomes

$$C_{+}C_{-} = 4\pi^{2}\left(Q^{2} - \sigma_{r}\right)^{2}\frac{\left[\left(Q^{2} - \sigma_{r}\right) - \mathscr{M}^{2}\right]}{\left[4\left(Q^{2} - \sigma_{r}\right) - 3\mathscr{M}^{2}\right]}; \tag{34}$$

the product depends on mass parameter and charge parameter. Thus the product of specific heats is not universal.

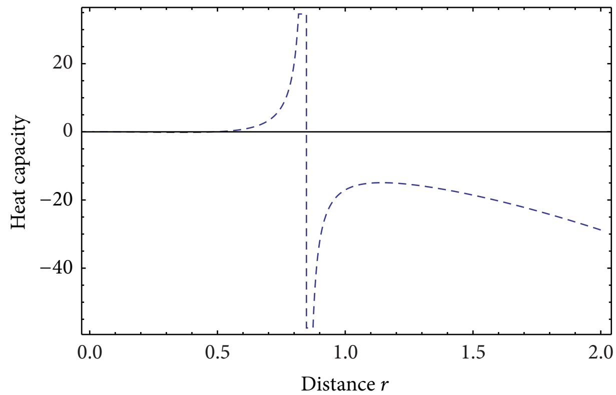

Figure 1: Heat capacity undergoes phase transition from instability to stability, diverges at $r = \sqrt{3(Q^{2} - \sigma_{r})}$, and again goes to instable region; we chose $\sigma_{r} = 0.01$ and $Q = 0.5$.

3. RN Black Hole Surrounded by Dust

Metric of RN black hole surrounded by dust is the same as in (1), $f(r)$ is defined in (2) with $\omega = 0$, and $\sigma = \sigma_{d}$ becomes

$$f(r) = 1 - \frac{2\mathscr{M}}{r} + \frac{Q^{2}}{r^{2}} - \frac{\sigma_{d}}{r}, \tag{35}$$

where $[\sigma_{d}] = L$. The horizons are

$$r_{\pm} = \frac{2\mathscr{M} + \sigma_{d} \pm \sqrt{\left(2\mathscr{M} + \sigma_{d}\right)^{2} - 4Q^{2}}}{2}. \tag{36}$$

Area of the horizons $\mathscr{H}^{\pm}$ is

$$\begin{aligned}\mathscr{A}_{\pm} &= \pi\left[2\left(2\mathscr{M} + \sigma_{d}\right)^{2} - 4Q^{2}\right.\\ &\quad\left.\pm 2\left(2\mathscr{M} + \sigma_{d}\right)\sqrt{\left(2\mathscr{M} + \sigma_{d}\right)^{2} - 4Q^{2}}\right]\\ &= 4\pi\left[\left(2\mathscr{M} + \sigma_{d}\right)r_{\pm} - Q^{2}\right].\end{aligned} \tag{37}$$

Entropy of the horizons is

$$\mathscr{S}_{\pm} = \pi\left[\left(2\mathscr{M} + \sigma_{d}\right)r_{\pm} - Q^{2}\right]. \tag{38}$$

Surface gravity and Hawking temperature of horizons are, respectively,

$$\kappa_{\pm} = \frac{2r_{\pm} - \left(2\mathscr{M} + \sigma_{d}\right)}{2\left[\left(2\mathscr{M} + \sigma_{d}\right)r_{\pm} - Q^{2}\right]}, \tag{39}$$

$$T_{\pm} = \frac{2r_{\pm} - \left(2\mathscr{M} + \sigma_{d}\right)}{4\pi\left[\left(2\mathscr{M} + \sigma_{d}\right)r_{\pm} - Q^{2}\right]} = \frac{1}{4\pi}\left(\frac{r_{\pm}^{2} - Q^{2}}{r_{+}^{3}}\right), \tag{40}$$

where we have used $r_{\pm}^{2} = (2\mathscr{M} + \sigma_{d})r_{\pm} - Q^{2}$. The Komar energy becomes

$$E_{\pm} = \frac{2r_{\pm} - \left(2\mathscr{M} + \sigma_{d}\right)}{2}. \tag{41}$$

Product of surface gravities and temperatures at the horizons is

$$\kappa_+\kappa_- = 4\pi^2 T_+ T_- = -\frac{(2\mathscr{M}+\sigma_d)^2 - 4Q^2}{4Q^4}. \tag{42}$$

Product of Komar energies at the horizons is

$$E_+E_- = \frac{4Q^2 - (2\mathscr{M}+\sigma_d)^2}{4}. \tag{43}$$

Note that all products are mass dependent, so these quantities are not universal. Products of areas and entropies at both horizons $\mathscr{H}^\pm$ are

$$\mathscr{A}_+\mathscr{A}_- = 16\mathscr{S}_+\mathscr{S}_- = 16\pi^2 Q^4. \tag{44}$$

It is clear that area product and entropy product are universal entities.

3.1. Smarr Formula for Cauchy Horizon ($\mathscr{H}^-$). Area of both horizons must be constant given by

$$\begin{aligned}\mathscr{A}_\pm = \pi \Big[& 2(2\mathscr{M}+\sigma_d)^2 - 4Q^2 \\ & \pm 2(2\mathscr{M}+\sigma_d)\sqrt{(2\mathscr{M}+\sigma_d)^2 - 4Q^2}\Big].\end{aligned} \tag{45}$$

Using (45) mass of the black hole or ADM mass is expressed in terms of the areas of horizons as

$$\mathscr{M}^2 + \mathscr{M}\sigma_d = \frac{\mathscr{A}_\pm}{16\pi} + \frac{\pi Q^4}{\mathscr{A}_\pm} - \frac{(\sigma_d^2 - 2Q^2)}{4}. \tag{46}$$

Differential of mass could be expressed in terms of physical invariants of the horizons:

$$d\mathscr{M} = \mathscr{T}_\pm d\mathscr{A}_\pm + \Phi_\pm dQ, \tag{47}$$

where

$$\begin{aligned}\mathscr{T}_\pm &= \frac{1}{(2\mathscr{M}+\sigma_d)}\left(\frac{1}{16\pi} - \frac{\pi Q^4}{\mathscr{A}_\pm^2}\right), \\ \Phi_\pm &= \frac{1}{(2\mathscr{M}+\sigma_d)}\left(Q + \frac{4\pi Q^3}{\mathscr{A}_\pm}\right).\end{aligned} \tag{48}$$

We can rewrite effective surface tension as

$$\begin{aligned}\mathscr{T}_\pm &= \frac{1}{16\pi(2\mathscr{M}+\sigma_d)}\left(1 - \frac{16\pi^2 Q^4}{\mathscr{A}_\pm^2}\right) \\ &= \frac{1}{8\pi(2\mathscr{M}+\sigma_d)}\Bigg[1 \\ &\quad - \frac{4(\mathscr{M}^2 + \mathscr{M}\sigma_d) + (\sigma_d^2 - 2Q^2)}{2r_\pm^2}\Bigg]\end{aligned} \tag{49}$$

or

$$\mathscr{T} = \pm\frac{\sqrt{(2\mathscr{M}+\sigma_d)^2 - 4Q^2}}{16\pi((2\mathscr{M}+\sigma_d)r_\pm - Q^2)} = \frac{\kappa_\pm}{8\pi}. \tag{50}$$

So the first law of black hole thermodynamics is verified, for RN black hole surrounded by dust, using the Smarr formula approach.

3.2. Christodoulou-Ruffini Mass Formula for RN Black Hole Surrounded by Dust. The expression for irreducible mass for RN black hole surrounded by dust is

$$\mathscr{M}_{\text{irr}\pm} = \frac{2\mathscr{M} + \sigma_d \pm \sqrt{(2\mathscr{M}+\sigma_d)^2 - 4Q^2}}{4}. \tag{51}$$

Here $\mathscr{M}_{\text{irr}-}$ and $\mathscr{M}_{\text{irr}+}$ are irreducible masses defined on inner and outer horizons, respectively. Area of $\mathscr{H}^\pm$, in terms of $\mathscr{M}_{\text{irr}\pm}$, is

$$\mathscr{A}_\pm = 16\pi(\mathscr{M}_{\text{irr}\pm})^2. \tag{52}$$

Product of the irreducible mass at the horizons $\mathscr{H}^\pm$ is

$$\mathscr{M}_{\text{irr}+}\mathscr{M}_{\text{irr}-} = \sqrt{\frac{\mathscr{A}_+\mathscr{A}_-}{(16\pi)^2}} = \frac{Q^2}{4}. \tag{53}$$

This product is independent of mass of the black hole. Mass of the black hole expressed in terms of its irreducible mass and charge is

$$\mathscr{M}^2 + \mathscr{M}\sigma_d = \frac{\rho_\pm^4 + Q^4}{4\rho_\pm^2} - \sigma_d + 2\mathscr{M}_{\text{irr}+}\mathscr{M}_{\text{irr}-}. \tag{54}$$

3.3. Heat Capacity $C_\pm$ on $\mathscr{H}^\pm$. Mass of RN black hole surrounded by dust in terms of $r_\pm$ is

$$\mathscr{M} = \frac{r_\pm^2 + Q^2 - \sigma_d r_\pm}{2r_\pm}. \tag{55}$$

Partial derivatives of mass $\mathscr{M}$ and temperature $T_\pm$ with respect to $r_\pm$ are

$$\begin{aligned}\frac{\partial\mathscr{M}}{\partial r_\pm} &= \frac{r_\pm^2 - Q^2 + \sigma_d r_\pm}{2r_\pm^2}, \\ \frac{\partial T_\pm}{\partial r_\pm} &= \frac{1}{4\pi}\frac{(3Q^2 - r_\pm^2)}{r_\pm^4},\end{aligned} \tag{56}$$

where T is given in (40). The heat capacity $C = (\partial\mathscr{M}/\partial r_\pm)(\partial r_\pm/\partial T)$ at the horizons is

$$C_\pm = \frac{2\pi r_\pm^2(r_\pm^2 - Q^2 + \sigma_d r_\pm)}{r_\pm^2 - Q^2}. \tag{57}$$

In this case the product formula for heat capacity is found to be

$$C_+C_- = \frac{4\pi^2 Q^4\left[4Q^4 - Q^2(2\mathscr{M}+\sigma_d)^2 + \sigma_d^2 Q^2\right]}{4Q^4 - Q^2(2\mathscr{M}+\sigma_d)^2}. \tag{58}$$

It is clear that the product formula does depend on mass parameter, so it is not universal in nature. Note that there are two possible cases for heat capacity, C, to be positive.

Case 1. When both $r_\pm^2 - Q^2 + \sigma_d r_\pm$ and $r^2 - Q^2$ are positive.

Case 2. When both $r_\pm^2 - Q^2 + \sigma_d r_\pm$ and $r^2 - Q^2$ are negative.

Considering $\sigma_d r_\pm$ as a positive quantity (for physically accepted region, r), from Case 1, we can say that C is positive for only $r_\pm^2 - Q^2 > 0$; that is,

$$\sqrt{Q^2} < r < -\sqrt{Q^2}, \tag{59}$$

while from Case 2 we can say that C is negative for only $r_\pm^2 - Q^2 + \sigma_d r_\pm < 0$; that is,

$$r < \frac{-\sigma_d}{2} + \frac{1}{2}\sqrt{4Q^2 + \sigma_d^2}. \tag{60}$$

Heat capacity is negative if the following cases hold.

Case a. $r_\pm^2 - Q^2 + \sigma_d r_\pm > 0$ and $r_\pm^2 - Q^2 < 0$.

Case b. $r_\pm^2 - Q^2 + \sigma_d r_\pm < 0$ and $r_\pm^2 - Q^2 > 0$.

Case b is not possible mathematically since $\sigma_d r_\pm > 0$, while in Case a heat capacity is negative in the region, where r satisfies both of the following conditions:

$$\begin{gathered} \frac{-\sigma_d}{2} + \frac{1}{2}\sqrt{4Q^2 + \sigma_d^2} < r < \frac{-\sigma_d}{2} - \frac{1}{2}\sqrt{4Q^2 + \sigma_d^2}, \\ -\sqrt{Q^2} < r < \sqrt{Q^2}, \end{gathered} \tag{61}$$

simultaneously. We consider that both σ and Q are positive in all the calculations. The behavior of the heat capacity given in (57) is shown in Figure 2.

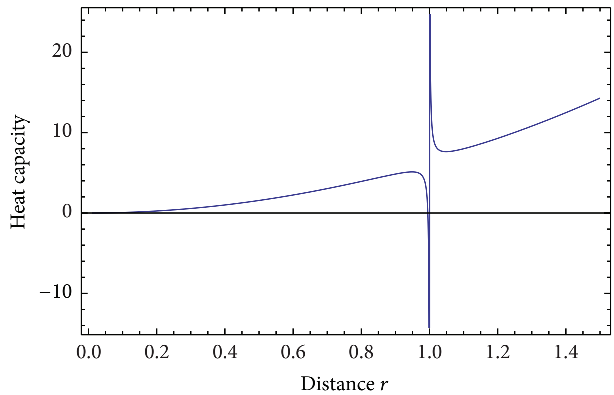

Figure 2: Heat capacity undergoes phase transition from stability to instability region; we chose $\sigma_d = 0.01$, $Q = 0.5$, and $\mathscr{M} = 1$.

4. Schwarzschild Black Hole Surrounded by Quintessence

Metric for Schwarzschild black hole surrounded by quintessence is the same as that defined in (1) and $f(r)$ defined in (2) with $\omega = -2/3$, $\sigma = \sigma_q$, and $Q = 0$ becomes

$$f(r) = 1 - \frac{2\mathscr{M}}{r} - \sigma_q r, \tag{62}$$

where dimensions of σ_q are that of L^{-1}. The horizons, $r_\pm$, of the black hole are

$$r_\pm = \frac{1 \pm \sqrt{1 - 8\mathscr{M}\sigma_q}}{2\sigma_q}. \tag{63}$$

Product of the two horizons yields

$$r_+ r_- = \frac{2\mathscr{M}}{\sigma_q}, \tag{64}$$

and it is depending on mass of the black hole and σ_q. Areas of the horizons are

$$\mathscr{A}_\pm = 4\pi\left[\frac{r_\pm - 2\mathscr{M}}{\sigma_q}\right]. \tag{65}$$

Entropy at the horizons $\mathscr{H}^\pm$ is

$$\mathscr{S}_\pm = \frac{\pi}{\sigma_q}\left(r_\pm - 2\mathscr{M}\right). \tag{66}$$

Hawking temperature of the horizons is

$$T_\pm = \frac{1}{4\pi}\left[\frac{1 - 2\sigma_q r_\pm}{r_\pm}\right], \tag{67}$$

and the surface gravity on the black hole horizons $\mathscr{H}^\pm$ is given by

$$\kappa_\pm = \frac{1}{2}\left[\frac{1 - 2\sigma_q r_\pm}{r_\pm}\right]. \tag{68}$$

The Komar energy is given by

$$E_\pm = \frac{2\mathscr{M}\left(1 + 2\sigma_q r_\pm\right) - r_\pm}{2\sigma_q r_\pm}. \tag{69}$$

Product of surface gravities and temperatures of $\mathscr{H}^\pm$ is

$$\kappa_+\kappa_- = 4\pi^2 T_+ T_- = \frac{\left(8\mathscr{M}\sigma_q - 1\right)\sigma_q}{8\mathscr{M}}. \tag{70}$$

Product of Komar energies of the horizons is

$$E_+ E_- = \frac{\mathscr{M}\left(8\mathscr{M}\sigma_q - 1\right)}{2\sigma_q}, \tag{71}$$

respectively. It is clear that all these products are depending on the mass of the black hole, so these quantities are not universal. The products of areas and entropies at $\mathscr{H}^\pm$ are

$$\mathscr{A}_+\mathscr{A}_- = 16\mathscr{S}_+\mathscr{S}_- = \left(\frac{8\pi\mathscr{M}}{\sigma_q}\right)^2; \tag{72}$$

again both products are not universal quantities.

4.1. Smarr Formula for Cauchy Horizon ($\mathscr{H}^-$). Write area of both horizons of the black hole as

$$\mathscr{A}_\pm = \frac{\pi}{\sigma_q^2}\left[2 - 8\mathscr{M}\sigma_q \pm 2\sqrt{1 - 8\mathscr{M}\sigma_q}\right]. \tag{73}$$

Using (73) mass of the black hole or ADM mass is expressed in terms of the areas of horizons as

$$4\mathscr{M}^2 + \frac{\mathscr{M}\mathscr{A}\sigma_q}{\pi} = \frac{1}{16\pi}\left[4\mathscr{A}_\pm - \frac{\mathscr{A}_\pm^2\sigma_q^2}{\pi}\right]. \quad (74)$$

Differential of mass, expressed in terms of physical invariants of the horizons, is

$$d\mathscr{M} = \mathscr{T}_\pm d\mathscr{A}_\pm , \quad (75)$$

where

$$\begin{aligned}\mathscr{T}_\pm &= \frac{1}{8\pi\mathscr{M} + \sigma_q\mathscr{A}_\pm}\left[\frac{4\pi - 16\pi\sigma_q\mathscr{M} - 2\mathscr{A}_\pm\sigma_q^2}{16\pi}\right] \\ &= \frac{1}{16\pi\mathscr{A}_\pm}\left[8\pi\mathscr{M} - \mathscr{A}_\pm\sigma_q\right] = \frac{\kappa_\pm}{8\pi},\end{aligned} \quad (76)$$

where we have used $\mathscr{M} = (r_\pm - \sigma_q r_\pm^2)/2$ and κ is defined in (68). Hence the first law of thermodynamics is satisfied by Schwarzschild black hole surrounded by quintessence.

4.2. Christodoulou-Ruffini Mass Formula for Schwarzschild Black Hole Surrounded by Quintessence. The irreducible mass of Schwarzschild black hole surrounded by quintessence is

$$\mathscr{M}_{\mathrm{irr}\pm} = \frac{1 \pm \sqrt{1 - 8\mathscr{M}\sigma_q}}{4\sigma_q}. \quad (77)$$

Here $\mathscr{M}_{\mathrm{irr}-}$ and $\mathscr{M}_{\mathrm{irr}+}$ are irreducible masses defined on inner and outer horizons, respectively. Area of $\mathscr{H}^\pm$, in terms of $\mathscr{M}_{\mathrm{irr}\pm}$, is

$$\mathscr{A}_\pm = 16\pi\left(\mathscr{M}_{\mathrm{irr}\pm}\right)^2. \quad (78)$$

Product of the irreducible mass at the horizons $\mathscr{H}^\pm$ is

$$\mathscr{M}_{\mathrm{irr}+}\mathscr{M}_{\mathrm{irr}-} = \sqrt{\frac{\mathscr{A}_+\mathscr{A}_-}{(16\pi)^2}} = \frac{\mathscr{M}}{2\sigma}. \quad (79)$$

This product is depending on mass of the black hole. Expression of mass given in (74), in terms of irreducible mass, becomes

$$\begin{aligned}&8\sigma^2\mathscr{M}_{\mathrm{irr}+}^2\mathscr{M}_{\mathrm{irr}-}^2 + 4\pi\rho_\pm^2\left(\mathscr{M}_{\mathrm{irr}+}\mathscr{M}_{\mathrm{irr}-}\right) \\ &\quad = 4\rho_\pm^2\left(1 - 4\sigma^2\right).\end{aligned} \quad (80)$$

4.3. Heat Capacity $C_\pm$ on $\mathscr{H}^\pm$. The mass of Schwarzschild black hole surrounded by quintessence in terms of $r_\pm$ is

$$\mathscr{M} = \frac{r_\pm - \sigma_q r_\pm^2}{2}. \quad (81)$$

The partial derivative of mass $\mathscr{M}$ with respect to $r_\pm$ is

$$\frac{\partial\mathscr{M}}{\partial r_\pm} = \frac{1 - 2\sigma_q r_\pm}{2}, \quad (82)$$

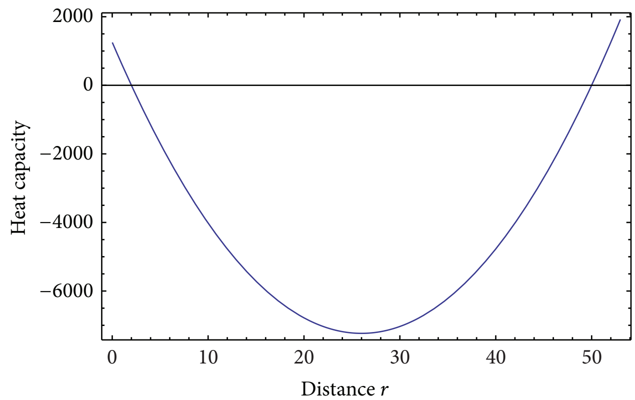

FIGURE 3: Heat capacity phase transition of Schwarzschild black hole surrounded by quintessence. We chose $\sigma_q = 0.01$ and $\mathscr{M} = 1$. Heat capacity is positive for $0 < r < 2$ and $r > 50$ and negative for $2 < r < 50$ and is zero at $r = 2$ and $r = 50$.

and using (67) we get

$$\frac{\partial T_\pm}{\partial r_\pm} = -\frac{1}{4\pi}\left[\frac{\sigma_q}{r_\pm^2 - 2\mathscr{M}}\right]. \quad (83)$$

The expression for heat capacity $C = (\partial\mathscr{M}/\partial r_\pm)(\partial r_\pm/\partial T)$ at the horizon becomes

$$C_\pm = \frac{-2\pi\left(1 - 2\sigma_q r_\pm\right)\left(r_\pm - 2\mathscr{M}\right)}{\sigma_q}. \quad (84)$$

Heat capacity would be positive if the following cases hold.

Case 1. $1 - 2\sigma_q r_\pm < 0$ and $r_\pm - 2\mathscr{M} > 0$.

Case 2. $1 - 2\sigma_q r_\pm > 0$ and $r_\pm - 2\mathscr{M} < 0$.

Heat capacity is negative if the following cases hold.

Case a. $1 - 2\sigma_q r_\pm > 0$ and $r_\pm - 2\mathscr{M} > 0$.

Case b. $1 - 2\sigma_q r_\pm < 0$ and $r_\pm - 2\mathscr{M} < 0$.

The behavior of the heat capacity given in (84) is shown in Figure 3 for $\mathscr{M} = 1$ and $\sigma_d = 0.01$; heat capacity is negative for $2 < r < 50$ and positive for $0 < r < 2$ and $r > 50$; it is zero at $r = 2$ and $r = 50$. A comparison of all the parameters calculated for Kiselev solutions is shown in Table 1.

5. Charged Dilaton Black Hole

The action for charged black hole in string theory is [31]

$$\mathscr{S} = \int d^4x\sqrt{-g}\left[-R + 2(\nabla\varphi)^2 + e^{-2a\varphi}F^2\right], \quad (85)$$

where F is the Maxwell field, φ is the scalar field, and a is an arbitrary parameter specifying the strength of dilaton and the Maxwell field's coupling. We are going to derive the area product formula and entropy product formula for

TABLE 1: A comparison of thermodynamical parameters for RN-radiation, RN-dust, and Schwarzschild-quintessence black holes.

Parameter	RN-radiation	RN-dust	Schwarzschild-quintessence
$r_\pm$	$\mathscr{M} \pm \sqrt{\mathscr{M}^{-}Q^2 + \sigma_r}$	$\dfrac{2\mathscr{M} + \sigma_d \pm \sqrt{(2\mathscr{M} + \sigma_d)^2 - 4Q^2}}{2}$	$\dfrac{1 \pm \sqrt{1 - 8\mathscr{M}\sigma_q}}{2\sigma_q}$
$\mathscr{A}_\pm$	$4\pi\left(2\mathscr{M}r_\pm - Q^2 + \sigma_r\right)$	$4\pi\left[\left(2\mathscr{M} + \sigma_d\right)r_\pm - Q^2\right]$	$4\pi\left[\dfrac{r_\pm - 2\mathscr{M}}{\sigma_q}\right]$
$S_\pm$	$\pi\left(2\mathscr{M}r_\pm - Q^2 + \sigma_r\right)$	$\pi\left[\left(2\mathscr{M} + \sigma_d\right)r_\pm - Q^2\right]$	$\dfrac{\pi}{\sigma_q}\left(r_\pm - 2\mathscr{M}\right)$
$T_\pm$	$\dfrac{r_\pm - \mathscr{M}}{2\pi\left(2\mathscr{M}r_\pm - Q^2 + \sigma_r\right)}$	$\dfrac{2r_\pm - \left(2\mathscr{M} + \sigma_d\right)}{4\pi\left[\left(2\mathscr{M} + \sigma_d\right)r_\pm - Q^2\right]}$	$\dfrac{1}{4\pi}\left[\dfrac{1 - 2\sigma_q r_\pm}{r_\pm}\right]$
$\kappa_\pm$	$\dfrac{r_\pm - \mathscr{M}}{\left(2\mathscr{M}r_\pm - Q^2 + \sigma_r\right)}$	$\dfrac{2r_\pm - \left(2\mathscr{M} + \sigma_d\right)}{2\left[\left(2\mathscr{M} + \sigma_d\right)r_\pm - Q^2\right]}$	$\dfrac{1}{2}\left[\dfrac{1 - 2\sigma_q r_\pm}{r_\pm}\right]$
$E_\pm$	$r_\pm - \mathscr{M}$	$\dfrac{2r_\pm - \left(2\mathscr{M} + \sigma_d\right)}{2}$	$\dfrac{2\mathscr{M}\left(1 + 2\sigma_q r_\pm\right) - r_\pm}{2\sigma_q r_\pm}$
$\kappa_+\kappa_-$	$\dfrac{Q^2 - \sigma_r - \mathscr{M}^2}{\left(Q^2 - \sigma_r\right)^2}$	$-\dfrac{\left(2\mathscr{M} + \sigma_d\right)^2 - 4Q^2}{4Q^4}$	$\dfrac{\left(8\mathscr{M}\sigma_q - 1\right)\sigma_q}{8\mathscr{M}}$
T_+T_-	$\dfrac{Q^2 - \mathscr{M}^2 - \sigma_r}{4\pi^2\left(Q^2 - \sigma_r\right)^2}$	$\dfrac{4Q^2 - \left(2\mathscr{M} + \sigma_d\right)^2}{16\pi^2 Q^4}$	$\dfrac{\left(8\mathscr{M}\sigma_q - 1\right)\sigma_q}{32\pi^2\mathscr{M}}$
E_+E_-	$Q^2 - \mathscr{M}^2 - \sigma_r$	$\dfrac{4Q^2 - \left(2\mathscr{M} + \sigma_d\right)^2}{4}$	$\dfrac{\mathscr{M}\left(8\mathscr{M}\sigma_q - 1\right)}{2\sigma_q}$
$\mathscr{A}_+\mathscr{A}_-$	$16\pi^2\left(Q^2 - \sigma_r\right)^2$	$16\pi^2 Q^4$	$\left(\dfrac{8\pi\mathscr{M}}{\sigma_q}\right)^2$
$\mathscr{S}_+\mathscr{S}_-$	$\pi^2\left(Q^2 - \sigma_r\right)^2$	$\pi^2 Q^4$	$\left(\dfrac{2\pi\mathscr{M}}{\sigma_q}\right)^2$
$\mathscr{M}_{\text{irr}+}\mathscr{M}_{\text{irr}-}$	$\dfrac{Q^2 - \sigma_r}{4}$	$\dfrac{Q^2}{4}$	$\dfrac{\mathscr{M}}{2\sigma}$
$C_\pm$	$\dfrac{2\pi r_\pm^2\left(r_\pm^2 - Q^2 + \sigma_r\right)}{3Q^2 - 3\sigma_r - r_\pm^2}$	$\dfrac{2\pi r_\pm^2\left(r_\pm^2 - Q^2 + \sigma_d r_\pm\right)}{r_\pm^2 - Q^2}$	$\dfrac{-2\pi\left(1 - 2\sigma_q r_\pm\right)\left(r_\pm - 2\mathscr{M}\right)}{\sigma_q}$
C_+C_-	$4\pi^2\left(Q^2 - \sigma_r\right)^2 \dfrac{\left[\left(Q^2 - \sigma_r\right) - \mathscr{M}^2\right]}{\left[4\left(Q^2 - \sigma_r\right) - 3\mathscr{M}^2\right]}$	$\dfrac{4\pi^2 Q^4\left[4Q^4 - Q^2\left(2\mathscr{M} + \sigma_d\right)^2 + \sigma_d^2 Q^2\right]}{4Q^4 - Q^2\left(2\mathscr{M} + \sigma_d\right)^2}$	$\dfrac{16\pi^2\mathscr{M}^2\left(8\mathscr{M}\sigma - 1\right)}{\sigma^2}$

a spherically symmetric dilaton black hole [31] whose metric can be written in Schwarzschild-like coordinates as

$$
\begin{aligned}
ds^2 = {} & -\mathscr{N}(r)\,dt^2 + \frac{dr^2}{\mathscr{N}(r)} \\
& + \mathscr{R}(r)^2\left(d\theta^2 + \sin^2\theta d\phi^2\right),
\end{aligned}
\tag{86}
$$

where the function $\mathscr{N}(r)$ is defined by

$$
\begin{aligned}
\mathscr{N}(r) &= \left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)^{(1-a^2)/(1+a^2)}, \\
\mathscr{R}^2(r) &= r^2\left(1 - \frac{r_-}{r}\right)^{2a^2/(1+a^2)}.
\end{aligned}
\tag{87}
$$

In these equations, r_+ and r_- are constants, which are related to mass and charge of the black hole as

$$
\begin{aligned}
\mathscr{M} &= \frac{r_+}{2} + \left(\frac{1 - a^2}{1 + a^2}\right)\frac{r_-}{2}, \\
Q &= \sqrt{\frac{r_+ r_-}{1 + a^2}},
\end{aligned}
\tag{88}
$$

where as usual $\mathscr{M}$ is mass of the black hole and Q is electric charge of the black hole. It may be noted that Q and a are positive. The horizons of the black hole are determined by the function $\mathscr{N}(r) = 0$ which yields

$$
\begin{aligned}
r_+ &= \mathscr{M} + \sqrt{\mathscr{M}^2 - \left(\frac{2n}{1 + n}\right)Q^2}, \\
r_- &= \frac{1}{n}\left[\mathscr{M} + \sqrt{\mathscr{M}^2 - \left(\frac{2n}{1 + n}\right)Q^2}\right],
\end{aligned}
\tag{89}
$$

and n is defined by

$$
n = \frac{1 - a^2}{1 + a^2}. \tag{90}
$$

Here r_+ and r_- are called event horizon ($\mathscr{H}^+$) or outer horizon and Cauchy horizon ($\mathscr{H}^-$) or inner horizon, respectively, and $r_+ = r_-$ or $\mathscr{M}^2 = ((1 + n)/2)Q^2$ corresponding to the extreme charged dilaton black hole.

Case I. When $a = 0$ or $n = 1$, the metric corresponds to RN black hole.

Case II. When $a = 1$ or $n = 0$, the metric corresponds to Gibbons-Maeda-Garfinkle-Horowitz-Strominger (GMGHS) black hole. The expressions for surface gravity of dilaton black hole at both horizons ($\mathscr{H}^{\pm}$) are

$$\kappa_{+} = \frac{1}{2r_{+}}\left(\frac{r_{+}-r_{-}}{r_{+}}\right)^{n}, \quad \kappa_{-} = 0. \tag{91}$$

The black hole temperature or Hawking temperature at $\mathscr{H}^{\pm}$ is

$$T_{+} = \frac{1}{4\pi r_{+}}\left(\frac{r_{+}-r_{-}}{r_{+}}\right)^{n}, \quad T_{-} = 0. \tag{92}$$

Areas of the horizons ($\mathscr{H}^{\pm}$) are

$$\mathscr{A}_{+} = 4\pi r_{+}^{2}\left(\frac{r_{+}-r_{-}}{r_{+}}\right)^{1-n}, \quad \mathscr{A}_{-} = 0. \tag{93}$$

Interestingly, the area of both horizons goes to zero at the extremal limit ($r_{+} = r_{-}$) which is quite different from the well-known RN and Schwarzschild black hole. The other characteristic of this spacetime is that there is a curvature singularity at $r = r_{-}$.

Now the entropies of both horizons ($\mathscr{H}^{\pm}$) are

$$\mathscr{S}_{+} = \pi r_{+}^{2}\left(\frac{r_{+}-r_{-}}{r_{+}}\right)^{1-n}, \quad \mathscr{S}_{-} = 0. \tag{94}$$

Finally, the Komar energy is given by

$$E_{+} = \frac{r_{+}-r_{-}}{2}, \quad E_{-} = 0. \tag{95}$$

Now we compute products of all the parameters given above:

$$\mathscr{A}_{+}\mathscr{A}_{-} = 0, \quad \mathscr{S}_{+}\mathscr{S}_{-} = 0, \quad \kappa_{+}\kappa_{-} = 0, \quad T_{+}T_{-} = 0, \quad E_{+}E_{-} = 0. \tag{96}$$

Interestingly their products go to zero value and independent of mass; thus they are universal quantities. All of the above thermodynamical quantities must satisfy the first law of thermodynamics:

$$d\mathscr{M} = \mathscr{T}_{\pm}d\mathscr{A}_{\pm} + \Phi_{\pm}dQ, \tag{97}$$

where

$$\Phi_{\pm} = \left(\frac{2n}{1+n}\right)\frac{Q}{r_{\pm}}. \tag{98}$$

The irreducible mass at $\mathscr{H}^{\pm}$ for this black hole is

$$\mathscr{M}_{\mathrm{irr}+} = \frac{r_{+}}{2}\left(\frac{r_{+}-r_{-}}{r_{+}}\right)^{(1-n)/2}, \quad \mathscr{M}_{\mathrm{irr}-} = 0. \tag{99}$$

Their product yields

$$\mathscr{M}_{\mathrm{irr}+}\mathscr{M}_{\mathrm{irr}-} = 0. \tag{100}$$

The heat capacity for this dilaton black hole is calculated to be

$$C_{+} = -2\pi r_{+}^{2}\frac{\left[1-(2n/(1+n))\left(Q^{2}/r_{+}\right)\right]\left[1-(2/(1+n))\left(Q^{2}/r_{+}\right)\right]}{\left[1-(2(1+2n)/(1+n))\left(Q^{2}/r_{+}\right)\right]\left[1-(2/(1+n))\left(Q^{2}/r_{+}\right)\right]^{n}}. \tag{101}$$

Due to curvature singularity at $r = r_{-}$ the heat capacity at the Cauchy horizon diverges. Thus the product of heat capacity diverges. Note that for odd n, heat capacity would be positive if

$$r < \frac{2(1+2n)Q^{2}}{1+n}, \quad r > \frac{2nQ^{2}}{1+n}, \tag{102}$$

or

$$r > \frac{2(1+2n)Q^{2}}{1+n}, \quad r < \frac{2nQ^{2}}{1+n}, \tag{103}$$

and heat capacity is negative for

$$r > \frac{2(1+2n)Q^{2}}{1+n}, \quad r > \frac{2nQ^{2}}{1+n}, \tag{104}$$

or

$$r < \frac{2(1+2n)Q^{2}}{1+n}, \quad r < \frac{2nQ^{2}}{1+n}. \tag{105}$$

The behavior of the heat capacity is shown in Figure 4. For $Q = 0.5$ and $n = 3$, the given expression of heat capacity (101)

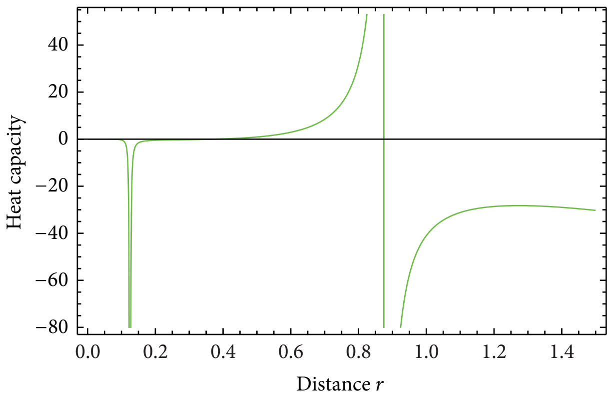

FIGURE 4: Heat capacity undergoes phase transition from instability to stability region and again to instability region, with divergence at $r = 2Q^2/(1+n)$ and $2(1+n)Q^2/(1+n)$; we chose $Q = 0.5$, $n = 3$, and $\mathscr{M} = 1$.

diverges at $r = 0.125$ and $r = 0.875$; it is positive for $0.375 < r < 0.875$ and negative for $0 < r < 0.125$, $0.125 < r < 0.375$, and $0.875 < r < 1.5$ (for $r \in (0, 1.5)$).

6. Conclusion

We have studied the thermodynamical properties on the inner and outer horizons of Kiselev solutions (RN black hole surrounded by energy-matter (radiation and dust) and Schwarzschild black hole surrounded by quintessence) and charged dilaton black hole. We have studied some important parameters of black hole thermodynamics with reference to their event and Cauchy horizons. We derive the expressions for temperatures and heat capacities of all the black holes mentioned above. It is observed that the product of surface gravities, surface temperature product, and product of Komar energies at the horizons are not universal quantities for the Kiselev's solutions while products of areas and entropies at both horizons are independent of mass of black hole (except for Schwarzschild black hole surrounded by quintessence). For dilaton black hole these products are universal except the products of specific heat which has shown divergent properties due to the curvature singularity at the Cauchy horizon. Thus the implication of these thermodynamical products may somehow give us further understanding of the microscopic nature of black hole entropy (both exterior and interior) in the black hole physics.

Using the heat capacity expressions, stability regions of the black holes are also observed graphically. Figures 1–4 show that the above-mentioned black holes undergo a phase transition under certain conditions on r. For RN black hole surrounded by radiation and dust and Schwarzschild black hole surrounded by quintessence, we derived the first law of thermodynamics using the Smarr formula approach. It is observed that the third law of thermodynamics which states that "surface gravity, κ, of a black hole cannot be reduced to zero in a finite sequence of processes" holds for all the above-mentioned black holes. The derived expressions of κ show that it is zero for extreme black holes only.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] D. Christodoulou, "Reversible and irreversible transformations in black-hole physics," *Physical Review Letters*, vol. 25, no. 22, article 1596, 1970.

[2] D. Christodoulou and R. Ruffini, "Reversible transformations of a charged black hole," *Physical Review D*, vol. 4, no. 12, pp. 3552–3555, 1971.

[3] R. Penrose and R. M. Floyd, "Extraction of rotational energy from a black hole," *Nature*, vol. 229, no. 6, pp. 177–179, 1971.

[4] S. W. Hawking, "Gravitational radiation from colliding black holes," *Physical Review Letters*, vol. 26, no. 21, pp. 1344–1346, 1971.

[5] J. D. Bekenstein, "Black holes and entropy," *Physical Review D*, vol. 7, pp. 2333–2346, 1973.

[6] J. D. Bekenstein, "Generalized second law of thermodynamics in black-hole physics," *Physical Review D*, vol. 9, no. 12, pp. 3292–3300, 1974.

[7] M. Jamil and M. Akbar, "Generalized second law of thermodynamics for a phantom energy accreting BTZ black hole," *General Relativity and Gravitation*, vol. 43, no. 4, pp. 1061–1068, 2011.

[8] M. Jamil, I. Hussain, and M. U. Farooq, "Thermodynamics of phantom energy in the presence of a Reissner-Nordström black hole," *Astrophysics and Space Science*, vol. 335, no. 2, pp. 339–343, 2011.

[9] M. Jamil and I. Hussain, "Accretion of phantom energy and generalized second law of thermodynamics for Einstein-Maxwell-Gauss-Bonnet black hole," *International Journal of Theoretical Physics*, vol. 50, no. 2, pp. 465–472, 2011.

[10] M. Akbar and A. A. Siddiqui, "Charged rotating BTZ black hole and thermodynamic behavior of field equations at its horizon," *Physics Letters B*, vol. 656, no. 4-5, pp. 217–220, 2007.

[11] A. Castro and M. J. Rodriguez, "Universal properties and the first law of black hole inner mechanics," *Physical Review D*, vol. 86, no. 2, Article ID 024008, 2012.

[12] J. Wang, W. Xu, and X.-H. Meng, "The 'universal property' of horizon entropy sum of black holes in four dimensional asymptotical (anti-)de-Sitter spacetime background," *Journal of High Energy Physics*, vol. 2014, no. 1, article 031, 2014.

[13] A. Curir, "Spin entropy of a rotating black hole," *Il Nuovo Cimento B*, vol. 51, no. 2, pp. 262–266, 1979.

[14] D. Pavón, "Phase transition in Reissner-Nordström black holes," *Physical Review D*, vol. 43, no. 8, pp. 2495–2497, 1991.

[15] S. W. Hawking and D. N. Page, "Thermodynamics of black holes in anti-de Sitter space," *Communications in Mathematical Physics*, vol. 87, no. 4, pp. 577–588, 1983.

[16] E. T. Newman, E. Couch, K. Chinnapared, A. Exton, A. Prakash, and R. Torrence, "Metric of a rotating, charged mass," *Journal of Mathematical Physics*, vol. 6, pp. 918–919, 1965.

[17] P. Pradhan, "Black hole interior mass formula," *The European Physical Journal C*, vol. 74, article 2887, 2014.

[18] P. Pradhan, "Thermodynamic products for sen black hole," http://arxiv.org/abs/1503.04514.

[19] M. Cvetič, G. W. Gibbons, and C. N. Pope, "Universal area product formulas for rotating and charged black holes in four

and higher dimensions," *Physical Review Letters*, vol. 106, no. 12, Article ID 121301, 2011.

[20] M. Visser, "Area products for stationary black hole horizons," *Physical Review D*, vol. 88, no. 4, Article ID 044014, 8 pages, 2013.

[21] V. V. Kiselev, "Quintessence and black holes," *Classical and Quantum Gravity*, vol. 20, no. 6, pp. 1187–1197, 2003.

[22] M. Jamil, S. Hussain, and B. Majeed, "Dynamics of particles around a Schwarzschild-like black hole in the presence of quintessence and magnetic field," *The European Physical Journal C*, vol. 75, article 24, 2015.

[23] S. Chandrasekhar, *The Mathematical Theory of Black Holes*, Oxford, UK, The Clarendon Press, 1983.

[24] S. W. Hawking, "Gravitational Radiation from Colliding Black Holes," *Physical Review Letters*, vol. 26, no. 21, pp. 1344–1346, 1971.

[25] E. Poisson, "An advance course in general relativity," Draft, 2002.

[26] A. Komar, "Covariant conservation laws in general relativity," *Physical Review*, vol. 113, pp. 934–936, 1959.

[27] L. Smarr, "Mass formula for kerr black holes," *Physical Review Letters*, vol. 30, no. 2, pp. 71–73, 1973.

[28] L. Smarr, "Surface geometry of charged rotating black holes," *Physical Review D*, vol. 7, no. 2, pp. 289–295, 1973.

[29] J. M. Bardeen, B. Carter, and S. W. Hawking, "The four laws of black hole mechanics," *Communications in Mathematical Physics*, vol. 31, pp. 161–170, 1973.

[30] D. J. Gross, M. J. Perry, and L. G. Yaffe, "Instability of flat space at finite temperature," *Physical Review D*, vol. 25, no. 2, pp. 330–355, 1982.

[31] D. Garfinkle, G. T. Horowitz, and A. Strominger, "Charged black holes in string theory," *Physical Review D*, vol. 43, no. 10, pp. 3140–3143, 1991.


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